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# CONTINUUM AND FINITE-PLAYER NONCOOPERATIVE MODELS OF COMPETITION

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SOCIAL SCIENCE WORKING PAPER 418

February 1982

## ABSTRACT

The anonymous interaction of large numbers of economic agents is a kind of noncooperative situation which is markedly different from small-numbers strategic conflict. The mathematical model of a nonatomic game, or a game with a continuum of players, has been introduced as a model for these many-agent situations on the basis that its equilibria should closely approximate those of games with large finite numbers of players. This paper contains a precise definition of what it means for a nonatomic game to be the limit of a sequence of finite-player games, and a theorem which states when the limit of equilibria of finite-player games will be an equilibrium of the nonatomic limit game. This is analogous to theorems prompted by Edgeworth's conjecture in core theory. It is derived from a general set of sufficient conditions for the graph of a noncooperative equilibrium correspondence to be closed.

# CONTINUUM AND FINITE-PLAYER NONCOOPERATIVE MODELS OF COMPETITION

Edward J. Green*

## 1. INTRODUCTION

Noncooperative game theory is an attempt to explain and analyze behavior in two kinds of situations which are markedly different from one another. The first kind of situation consists of conflicts among a small group of agents, each of whom can make unilateral decisions which may significantly affect the welfare of the others as well as his own welfare. Card games with high stakes and battles between opposing generals are canonical examples of such conflict. The second kind of situation is characterized by the individualistic but not deliberately adversary behavior of a large number of agents, none of whom alone is able to affect the circumstances of anyone except himself but whose actions in the aggregate determine the environment in which all must live. The canonical examples of this latter sort of anonymous interaction are, of course, competitive markets.

Noncooperative game theory has been largely a theory of games having a fixed finite set of players. Regardless of its suitability for the representation of small-numbers conflict, this emphasis on enumerating all the players and their actions, one by one, makes the study of anonymous individualistic behavior awkward. Microeconomics is full of elegant and persuasive arguments about the behavior of representative firms and representative consumers in competitive markets in general, but it is an ambitious project to use finite-player games to show that a simple model of noncooperative exchange yields competitive outcomes when there are many traders (cf. [10]).

There is another representation of noncooperative games which is much more suitable than the finite-player representation as a basis for competitive theory. This is Schmeidler's [12] model of an anonymous noncooperative strategic-form game with a nonatomic measure space of players, which is a noncooperative analogue of the nonatomic model of the core of an exchange economy due to Aumann [1]. Just as Aumann's model provides a formal setting in which the core and the Walras equilibrium set exactly coincide, so Schmeidler's model provides a setting in which Walras equilibrium coincides with noncooperative equilibrium for a fairly wide class of games (cf. [3]). Moreover, it can be shown in this setting that the objections which may be brought against strategic form as a representation of small-numbers conflict are irrelevant to competitive situations (cf. [5]). Thus Schmeidler's model appears to be an excellent foundation for competitive theory.

There is a problem about the use of Schmeidler's model, though, which is that the noncooperative analogue of Edgeworth's conjecture may fail. Sequences of larger and larger finite games can be constructed such that (a) these sequences intuitively have limits which are nonatomic games having only Walrasian noncooperative equilibrium allocations but (b) there are sequences of noncooperative equilibrium allocations of the finite games which converge to highly noncompetitive allocations in the limit (cf. [5,11]). Given this phenomenon of "discontinuity at infinity," the assumption of a continuum of players might be suspect. Fortunately, however, the phenomenon is limited in scope. For particular classes of games, Dubey, Mas-Colell and Shubik [3] and Green [5] have shown that there are conditions under which the equilibria of a sequence of games in which a finite set of players is replicated will converge to equilibria of the nonatomic limit game. General theorems about this convergence of equilibria are set forth in the present paper. The immediate purpose of these theorems is to provide widely applicable and easily understandable and verifiable criteria for when a noncooperative model of competition may appropriately be studied in the continuum-of-players setting.

This work on convergence of equilibria will closely parallel the study of the corresponding problem in core theory. In particular, two devices which originated in that study will be adapted to noncooperative equilibrium here. One of these is the restatement of the original problem of convergence as a question about the upper hemicontinuity of an equilibrium correspondence defined on a topological space of games. This formulation is due to Kannai [8]. The other device is the use of a statistical description of strategy vectors in order to provide a dimension-free comparison between games with different numbers of players. This statistical treatment of allocations of an exchange economy is due to Hildenbrand and his associates, and is exposited in [7].



These two devices will be combined by the use of an abstract representation of a game in the spirit of Debreu [2], to be called a pseudogame. Both strategic-form games and their statistical images are examples of pseudogames. The introduction of pseudogames enables Theorem 1, which asserts that the equilibrium correspondence of a topological space has a closed graph, to be stated very easily and proved in a transparent way. With this fact in hand, it is a routine matter to state sufficient conditions for the correspondence to be upper hemicontinuous.

Theorem 2 will verify that the statistical image of a game is a faithful representation. That is, when the game and its image are both regarded as pseudogames, a joint probability measure on players' characteristics and strategies is an equilibrium of the image if and only if it is the statistical distribution of an equilibrium strategy vector of the original game. This theorem justifies the use of Hildenbrand's device as the basis on which to formulate Theorem 3, a specific version of the upper-hemicontinuity theorem for spaces of games in which the number of players may increase to infinity. The conclusion of this paper explains in detail how Theorem 3 may be applied to show the validity of Schmeidler's nonatomic model to represent competitive environments as noncooperative games.

## 2. STRATEGIC-FORM GAMES AND PSEUDOGAMES

The representation of noncooperative games to be studied here will be motivated by showing its close relation to the widely used strategic (or normal) form of a game with finitely many players. Let $T = \{1,\ldots,n\}$ be the set of players. There is a set S of strategies, and a strategy vector is an assignment of strategies to players. That is, the set V of strategy vectors is defined by $V = S^T$.

Typically each player t has only a subset $S_t$ of S available to him. Thus not every strategy vector in V is feasible. Rather, if $(S_t)_{t\varepsilon T}$ describe the strategies available to players in game g, then the set $F_g$ of feasible strategy vectors in g is defined by

$F_g = \prod_{t\varepsilon T} S_t$.

Each player has a strict preference relation among strategy vectors. The preferences of all players in game g can be described jointly by a relation $P_g \subseteq T \times V^2$. The interpretation of $(t,v',v) \varepsilon P_g$ is that player t strictly prefers strategy vector v' to strategy vector v. (N.B. Transitivity and so forth need not be assumed.)

Strategy vector $v = (s_1,\ldots,s_n)$ is a Nash equilibrium point of game g if (a) $v \varepsilon F_g$, and (b) for no player t and strategy $s' \varepsilon S_t$ would t strictly prefer to change his strategy from $s_t$ to s', given that the other players' strategies would remain fixed. Condition (b) may be restated in terms of a correspondence which describes the ability of players to change strategy vectors by revising their choices of strategies within their own strategy sets. Define



$A_g \subseteq T \times V^2$ by $A_g = \{(t,v',v) | \pi_t(v') \varepsilon S_t$ and $\forall t' \neq t\ \pi_{t'}(v') = \pi_{t'}(v)\}$, where $\pi_t: V \to S$ is projection onto the t component. (I.e., $\pi_t(v)$ is the strategy chosen by player t in strategy vector v.) Then condition (b) may be restated as (b') $A_g \cap P_g \cap (T \times V \times \{v\}) = \emptyset$. This completes the description of finite-player strategic form games.

The sets T, S, $(S_t)_{t\varepsilon T}$ and $P_g$ are the primitive entities of a strategic-form game. The sets V, $F_g$ and $A_g$ used to define Nash equilibrium are formed from Cartesian products of the primitive sets. A <u>pseudogame</u> will be defined by taking all of the sets and relations T, S, V, $P_g$, $A_g$ and $F_g$ to be primitive, along with a correspondence $J_g: V \to T \times S$ which specifies the strategies chosen by the various players in each strategy vector. Equilibrium will still be defined by (a) and (b'), but V, $A_g$, $F_g$ and J need not be of the form required in strategic form. In particular, V may be something other than the product space $S^T$. It will be evident that pseudogames are closely related to the abstract economies studied by Debreu [2].

There will be a minor technical departure of the precise definition of a pseudogame from the account just given. This departure is required to describe nonatomic games adequately because, if players preferences were to compare only strategy vectors defined on the basis of aggregate decisions and if no player is to have aggregate significance, then every strategy vector would necessarily be an equilibrium. In order to avoid this trivialization, a player's preferences will be specified to compare his circumstances: combinations of his own strategy and the strategy vector. The statistical viewpoint to be taken later will also focus attention on the combinations of players' characteristics and strategies rather than on the strategies of individual players. These characteristic-strategy combinations will be called decisions. The definitions of circumstance and decision are now presented and related to strategic form, after which the precise definition of pseudogames will be given.

A player's decision in a strategy vector is described by identifying the player and the strategy which he plays. In particular, the ordered pair $(t,\pi_t(v))$ is the decision of player t in strategic-form strategy vector v. Every decision is an element of the product set T x S. This set will be denoted by D. The correspondence $J:V \to D$ is defined by $J(v) = \{(t,\pi_t(v)) | t \in T\}$ for a strategic-form game.

The circumstance of a player in a strategy vector is described by the player's strategy and the strategy vector itself. That is, the ordered pair $(\pi_t(v),v)$ is the circumstance of player t in strategic-form strategy vector v. Every circumstance is an element of the product set S x V, which will be denoted by C. Players' preferences will compare circumstances rather than strategy vectors. To translate from the strategic-form to the pseudogame representation of a finite-player game, simply say that (s',v') is strictly preferred by player t to (s,v) in the pseudogame if and only if v' is strictly preferred to v in strategic form.



The precise definition of a pseudogame is now given. A pseudogame is defined in terms of a set T of player types, a set S of strategies, and a set V of strategy vectors. These sets will all be assumed to be topological spaces. A strategy choice by a player type is a decision. A strategy choice made in the context of a strategy vector is a circumstance. Formally, let D = T x S and C = S x V denote the set of decisions and the set of circumstances, respectively. These (and other product sets to be introduced) are topologized as product spaces.

Let g be a game. Each player type in g has a preference relation over circumstances. The preference relations of all player types are jointly described by a relation $P_g \subseteq T \times C^2$. The interpretation of $(t,c',c) \in P_g$ is that players of type t strictly prefer circumstance c' to circumstance c. However, no formal restrictions (e.g., that the preference relation of each type is transitive) are placed on $P_g$.

Players can change their circumstances by changing their decisions. Their ability to do this is described by an alternative correspondence $A_g:T \times C \to C$. That $(s',v') \in A_g(t,[s,v])$ means that, if a single player of type t were to change his strategy choice from s to s' when the strategy vector was v, that strategy vector would be changed to v'. A decision is inadmissable if the player making it has neglected a preferred alternative. Formally, define the inadmissible-decision correspondence $I_g:V \to D$ by

$$(t,s) \in I_g(v) \Leftrightarrow \exists c\ [c \in A_g(t,[s,v]) \text{ and } (t,c,[s,v]) \in P_g]. \qquad (1)$$

Each strategy vector results from a combination of strategy choices. These are specified by a correspondence $J_g:V \to D$. The interpretation of $(t,s) \in J_g(v)$ is that, in strategy vector v, at least one player of type t has chosen strategy s.

In order to compare several different games, we will have to embed their spaces of strategy vectors in a common space. Thus, for any particular game being considered, not every strategy vector in the common space V will be feasible. A subset $F_g \subseteq V$ specifies the feasible strategy vectors in game g.

An equilibrium point is a feasible strategy vector in which no inadmissible decisions are made. Formally, the equilibrium set $E_g \subseteq V$ of game g is defined by

$$v \in E_g \Leftrightarrow [v \in F_g \text{ and } I_g(v) \cap J_g(v) = \emptyset]. \qquad (2)$$

## 3. TOPOLOGICAL FAMILIES OF PSEUDOGAMES

Consider a topological space G, the points of which are pseudogames. Suppose that these are all defined on the same triple of spaces T, S and V. The characterization of pseudogames in G may be consolidated into a single relation $P \subseteq G \times T \times C^2$ and correspondences $F:G \to V$, $A:G \times T \times C \to C$ and $J:G \times V \to D$. Thus $(g,t,c',c) \in P \Leftrightarrow (t,c',c) \in P_g$, $v \in F(g) \Leftrightarrow v \in F_g$, and so forth. The inadmissible-decision correspondence $I:G \times V \to D$ is define by



$$(t,s) \in I(g,v) \Leftrightarrow \exists c\ [c \in A(g,t,[s,v]) \text{ and } (g,t,c,[s,v]) \in P]. \qquad (3)$$

The equilibrium correspondence $E:G \to V$ is defined by

$$v \in E(g) \Leftrightarrow [v \in F(g) \text{ and } I(g,v) \cap J(g,v) = \emptyset]. \qquad (4)$$

Clearly $v \in E(g) \Leftrightarrow v \in E_g$ by equations (1)-(4).

A <u>topological family</u> of pseudogames is specified by a relation P and correspondences F, A and J which satisfy

P is open, (5)

F is closed, (6)

A is lower hemicontinuous, and (7)

J is lower hemicontinuous (8)

## 4. THE EQUILIBRIUM CORRESPONDENCE OF A TOPOLOGICAL FAMILY

A correspondence $H:X \to Y$ is a mapping of the topological space X to subsets of the space Y. H is open or closed if its graph $\{(x,y)|y \in H(x)\}$ is open or closed, respectively. H is lower hemicontinuous (l.h.c.) if $\{x \mid H(x) \cap U \neq \emptyset\}$ is open in X for every open set U in Y. H is upper hemicontinuous (u.h.c.) if $\{x \mid H(x) \subseteq U\}$ is open in X for every open set U in Y and H(x) is nonempty for every x in X.

In this section it will be shown that the equilibrium correspondence of a topological family of pseudogames is always closed, and a sufficient condition will be established for such a correspondence to be upper hemicontinuous.

Lemma 1: Let X and Y be two arbitrary topological spaces. If $H:X \to Y$ is open and $K:X \to Y$ is l.h.c., then $W = \{x \mid H(x) \cap K(x) \neq \emptyset\}$ is open.

Proof: Suppose that $x \in W$ and that $y \in H(x) \cap K(x)$. Since H is open, there are neighborhoods U of x and Z of y such that $U \times Z \subseteq H$. Since K is l.h.c., the set $V = \{v \in X \mid K(v) \cap X \neq \emptyset\}$ is a neighborhood of x. W is open, then, because x (which is an arbitrary element of W) satisfies $x \in U \cap V \subseteq W$. Q.E.D.

Lemma 2: The inadmissible-decision correspondence of a topological family is open.

Proof: Define $H:G \times T \times C \to C$ by $c' \in H(g,t,c) \iff (g,t,c',c) \in P$. To apply Lemma 1, let K = A. By (5), (7) and Lemma 1, $W = \{(g,d,v) \mid H(g,d,v) \cap A(g,d,v) \neq \emptyset\}$ is open. Since $d \in I(g,v) \iff (g,d,v) \in W$, I is open. Q.E.D.



Theorem 1: The equilibrium correspondence of a topological family is closed.

Proof: It will be proved that $(G \times V)\setminus E$ is open. Define $W = \{(g,v) \mid I(g,v) \cap J(g,v) \neq \emptyset\}$. By (4), $(G \times V)\setminus E = W \cup (V\setminus F)$. Since F is closed by (6), it is sufficient to show that W is open. Letting H = I and K = J in Lemma 1, this is immediate by Lemma 2 and (8). Q.E.D.

Corollary: If E is nonempty valued and F is u.h.c. and compact valued, then E is u.h.c.

Proof: This is an immediate consequence of Theorem 1 and [7, Chapter 1, Proposition B.III.2]. Q.E.D.

Four examples are now provided which show that Theorem 1 would not be valid (i.e., that E might not be closed) if (5)-(8) were not required. In each example, T is an arbitrary space and $G = S = V = \mathbb{R}$. Z will denote $\{(r,r) \mid r \in \mathbb{R}\}$.

Example 1: P is not open. Define $P = \{(g,t,c,[s,v]) \mid v = g\}$, $F(g) = V$, $A(g,t,c) = C$, and $J(g,v) = T \times \{v\}$. Then $E = \mathbb{R}^2\setminus Z$.

Example 2: F is not closed. Define $P = \emptyset$, $F(g) = V\setminus\{g\}$, $A(g,t,c) = C$ and $J(g,v) = T \times \{v\}$. Then $E = \mathbb{R}^2\setminus Z$.

Example 3: A is not l.h.c. Define $P = G \times T \times C^2$, $F(g) = V$,

$$A(g,t,c) = \begin{cases} \{c\} & \text{if } g \neq 0 \\ C & \text{if } g = 0 \end{cases}, \text{ and } J(g,v) = T \times \{v\}. \text{ Then } E = \mathbb{R}^2\setminus(\{0\} \times \mathbb{R}).$$

<u>Example 4</u>: J is not l.h.c. Define $P = G \times T \times C^2$, $F(g) = V$, $A(g,t,c) = C$, and $J(g,v) = \begin{cases} \emptyset & \text{if } g \neq 0 \\ T\times\{v\} & \text{if } g = 0 \end{cases}$. Then $E = \mathbb{R}^2 \setminus (\{0\} \times \mathbb{R})$.

These pathologies can be exhibited also in topological families, the elements of which are derived from strategic-form games in which the players have rational preferences.

## 5. MEASURE-THEORETIC PRELIMINARIES

Games in strategic form (also called normal form by some authors) with a measure space of players will be introduced in the next section. These include the nonatomic games of Schmeidler [12]. For the remainder of the paper T and S will be assumed to be complete separable metric spaces, as will a set N of <u>players</u>. Some measure-theoretic preliminaries are now taken care of.

$\mathbb{R}$ denotes the real numbers. $\lambda$ denotes Lebesgue measure on $\mathbb{R}$.

The Borel $\sigma$-algebra of X is the smallest $\sigma$-algebra containing the open sets of X. A function $f:X \to Y$ is Borel measurable if the inverse image of every Borel subset of Y is a Borel subset of X. $Y^X$ will denote the set of Borel-measurable functions from X to Y. Set and function quantifiers range over Borel sets and over Borel-measurable functions, respectively.

A Borel measure on X is a nonnegative, real-valued, countably additive set function on the Borel $\sigma$-algebra of X. If m is a Borel measure on X and d is a metric on Y, then the topology on $Y^X$ of



convergence in m-measure is generated by the subbasic sets $U(f,q,r)$, where $f \in Y_X$, $q > 0$ and $r > 0$. $U(f,q,r)$ is defined by

$\forall f' \in Y^X\ f' \in U(f,q,r) \iff m(\{x \mid d(f(x),f'(x)) \geq q\}) < r$.

If X is a separable metric space, then M(X) will denote the set of Borel measures on X. <u>BY</u> [4, Exercise III.9.22], every $m \in M(X)$ is regular. I.e., for every $B \subseteq X$ and $r > 0$ there exist a closed $H \subseteq X$ and an open $U \subseteq X$ such that $H \subseteq B \subseteq U$ and $m(U) - m(H) < r$. It follows from [4, Theorem IV.6.2] that M(X) under the total-variation norm is isometrically embeddable in the dual space $C^*(X)$ of the Banach space of bounded, continuous, real-valued functions on X. If X is compact, then M(X) is isometrically isomorphic to the positive cone of $C^*(X)$ by [4, Theorem III.5.13].

M(X) will be regarded as a subspace of $C^*(X)$ under the weak* topology. This topology is generated by the subbasic open sets $W(f,q,r)$, where $f \in C(X)$, $q \in \mathbb{R}$ and $r > 0$. $W(f,q,r)$ is defined by $\forall m \in M(X)\ m \in W(f,q,r) \iff |\int_X f\,dm - q| < r$.

Suppose that X is separable. Then it is second countable. I.e., it has a countable base $\{W_1, W_2, \ldots\}$. For $m \in M(X)$, define $Z(m) = \{k \mid m(W_k) = 0\}$. The support of m is its image under the correspondence $\mathrm{supp}: M(X) \to X$ defined by $\mathrm{supp}(m) = X \setminus \bigcap_{k \in Z(m)} W_k$. Then $\mathrm{supp}(m)$ is closed, and $\forall B \subseteq X\ m(B) = m(B \cap \mathrm{supp}(m))$. Also, for every open $U \subseteq X$, $m(U) = 0 \iff U \cap \mathrm{supp}(m) = \emptyset$.

If X and Y are both separable metric spaces, then the Borel $\sigma$-algebra of $X \times Y$ is generated by product sets $B \times C$, where $B \subseteq X$ and

$C \subseteq Y$, by [9, Chapter 1, Theorem 1.10]. Thus, by [4, Theorem III.11.2], a unique measure m in $M(X \times Y)$ is specified by the equations $m(B \times C) = n(B) \cdot p(C)$ (where B and C are subsets of X and Y, respectively) if $n \in M(X)$ and $p \in M(Y)$.

For $B \subseteq X$, #B denotes the cardinality of B, and the characteristic function of B is denoted by $\chi_B : X \to \{0,1\}$. Free variables in formal statements implicitly are universally quantified over their appropriate domains. E.g., $v \in E(g) \iff v \in E_g$ should be read as $\forall v \in V \ \ \forall g \in G \ \ v \in E(g) \iff v \in E_g$.

## 6. GAMES IN STRATEGIC FORM WITH A MEASURE SPACE OF PLAYERS

A strategic-form game is defined on a player space N, endowed with a measure $n \in M(N)$, and on a strategy space S. Strategy vectors are functions in $S^N$, and are topologized by convergence in n-measure.

Players' preferences in game g are specified by an open relation $P_g \subseteq N \times (S \times S^N)^2$. A player's own action is made an explicit argument of his preference, although this is redundant (i.e., f(i) is the strategy of player i in strategy vector f), because the openness requirement would otherwise force nonatomic players to be indifferent to unilateral changes in their own strategies.

Feasibility in strategic form is defined directly in terms of players' strategy sets (rather than in terms of strategy vectors) by a closed and l.h.c. correspondence $F_g : N \to S$. $F_g(i)$ is the set of



strategies feasible for player i. The set $V_g$ of feasible strategy vectors in game g is defined to be the set of selections from $F_g$ (i.e., strategy vectors in which almost all players are assigned feasible strategies). I.e.,

$$f \in V_g \iff n(\{i \in N \mid f(i) \notin F_g(i)\}) = 0. \tag{9}$$

<u>Proposition 1</u>: If $f \in S^N$, then $\{i \in N \mid f(i) \notin F_g(i)\}$ is a Borel set.

<u>Proof</u>: Since $F_g$ is closed and since N and S are second countable, there is a sequence $(U_k, W_k)_{k=1,2,..}$ such that each $U_k$ and $W_k$ are open in N and S, respectively, and $(N \times S) \setminus F_g = \bigcup_{k=1}^{\infty} (U_k \times W_k)$. Hence

$\{i \in N \mid f(i) \notin F_g(i)\} = \bigcup_{k=1}^{\infty} (U_k \cap f^{-1}(W_k))$. Q.E.D.

Player i changes a strategy vector f by changing f(i). Rather than an alternative correspondence, in strategic form there is for each $i \in N$ an alternative function $a_i : S \times S^N \to S^N$, defined by

$$f' = a_i(s,f) \iff [f'(i) = s \text{ and } \forall j \neq i \ f'(j) = f(j)]. \tag{10}$$

In strategic form, it is sufficient for the inadmissible-decision correspondence to specify the players who make inadmissible decisions in a strategy vector, because the strategy vector itself specifies their strategies. Define the inadmissible-decision correspondence $I_g: S^N \to N$ of game g by

$$I_g(f) = \{i \mid \exists s \in F_g(i)\ (i,s,a_i(s,f),f(i),f) \in P_g\}. \tag{11}$$

Proposition 2: $I_g(f)$ is a Borel set.

Proof: Define $N_0 = \{i \mid n(\{i\}) = 0\}$. $N \setminus N_0$ is countable, so it is sufficient to prove that $I_g(f) \cap N_0$ is a Borel set. Note that, for $i \in N_0$, $a_i(s,f)$ is in exactly the same subbasic open sets of $S^N$ as f is. Thus for all $s \in S$ and $i \in N_0$, $(i,s,a_i(s,f),f(i),f) \in P_g \Leftrightarrow (i,s,f,f(i),f) \in P_g$, because $P_g$ is open. Furthermore, $\{(i,s,s') \mid (i,s,f,s',f) \in P_g\}$ is open. By second countability, this set is $\bigcup_{k=1}^{\infty}(X_k \times Y_k \times Z_k)$ for some sequence of open sets $X_k \subseteq N$, $Y_k \subseteq S$ and $Z_k \subseteq S$. Define $W_k = \{i \mid F_g(i) \cap Y_k \neq \emptyset\}$. $W_k$ is open because $F_g$ is l.h.c. $I_g(f) \cap N_0 = \{i \mid \exists s \in F_g(i)(i,s,f,f(i),f) \in P_g\} \cap N_0 = \bigcup_{k=1}^{\infty}(W_k \cap X_k \cap f^{-1}(Z_k)) \cap N_0$, which is a Borel set. Q.E.D.

An equilibrium point of the strategic-form game g is a strategy vector $f \in V_g$ such that $n(I_g(f)) = 0$.



## 7. ANONYMOUS GAMES

Much of economic theory concerns games in which the preference of each player depends only on his own decision and on aggregate or statistical information about the decisions of other players. These will be called anonymous games, and will be defined here as a class of games in strategic form. A space T of player types and a space S of strategies are given. There is a continuous mass-revealing function $b: T \to [0,1]$. The set of players is $\underline{N} = T \times [0,1]$. This set is endowed with a measure having the property that the mass of almost every player is accurately described by the mass-revealing function. The set $\underline{G} \subseteq M(\underline{N})$ of such measures is defined by

$$n \in \underline{G} \Leftrightarrow \forall i \in \operatorname{supp}(n)\ n(\{i\}) = b(\tau(i)), \tag{12}$$

where $\tau: \underline{N} \to T$ is projection onto T (i.e., $\tau(t,r) = t$).

Let $n \in \underline{G}$ be the measure of players in anonymous game g. To describe the relation of players' preferences in g to aggregate information, the statistical distribution of a strategy vector in $\underline{V} = S^{\underline{N}}$ is introduced. The function $\mu: \underline{G} \times \underline{V} \to M(D)$ is defined by

$$m \in \mu(n,f) \Leftrightarrow \forall X \subseteq T\ \forall Y \subseteq S\ m(X \times Y) = n(\tau^{-1}(X) \cap f^{-1}(Y)). \tag{13}$$

I.e., $\mu(n,f)$ is the statistical distribution of decisions in f relative to n.

A player's preference in g must depend only on his own strategy and on aggregate information, and his preference and strategy set must be functions of his type. That is, there must be a relation $\underline{P} \subseteq T \times (S \times M(D))^2$ which satisfies

$$(i,s',f',s,f) \in P_g \Leftrightarrow (\tau(i),s',\mu(n,f'),s,\mu(n,f)) \in \underline{P}\ , \qquad (14)$$

and there must be a correspondence $\underline{F}:T \to S$ which satisfies

$$s \in F_g(i) \Leftrightarrow s \in \underline{F}(\tau(i)). \qquad (15)$$

The equilibrium set $E_g$ of g is defined (as for strategic-form games in general) by

$$f \in E_g \Leftrightarrow [f \in V_g \text{ and } n(I_g(f)) = 0], \qquad (16)$$

where $V_g$ and $I_g$ are defined by (9) and (11), respectively.

Remark 1: An arbitrary strategic-form game h is equivalent to some anonymous game g. Suppose that the set of players in h is N, under Borel measure m. Define T = N x [0,1], and define b(i,q) = q for $i \in N$ and $q \in [0,1]$. Then define the injection $\nu:N \to \underline{N}$ by $\nu(i) = (i,m(\{i\}),0)$. The measure $n \in \underline{G}$ defined by $n(E) = m(\nu^{-1}(B))$ is the measure of players in g. $\underline{P}$ and $\underline{F}$ are defined in a natural way



from $P_h$ and $F_h$, respectively, and then $P_g$ and $F_g$ are defined by (14) and (15). This construction has the property that $f \in E_g$ if and only if $f \circ \nu \in E_h$.

This section concludes with a lemma which shows that the aggregate information about a strategy vector is sufficient to determine whether it is an equilibrium point of an anonymous game.

Lemma 3: If $\mu(n,f) = \mu(n,f') = m$, then $f \in E_g \Leftrightarrow f' \in E_g$.

Proof: Suppose that $f \notin E_g$. By (16), either $f \notin V_g$ or else $n(I_g(f)) > 0$. It will be shown that f' shares the difficulty in either case, so that $f' \notin E_g$.

Case 1: Assume that $f \notin V_g$. I.e., $n(\{i|f(i) \notin F_g(i)\}) > 0$. From the proof of Proposition 1 there must be open sets $U \subseteq \underline{N}$ and $W \subseteq S$ such that $n(U \cap f^{-1}(W)) > 0$ and $(U \times W) \cap F_g = \emptyset$. The image $\tau(U)$ is open because $\tau$ is a projection and U is open, and $(\tau^{-1}(\tau(U)) \times W) \cap F_g = \emptyset$ by (15). Note that, by (13), $m(\tau(U) \times W) = n(\tau^{-1}(\tau(U)) \cap f^{-1}(W)) \geq n(U \cap f^{-1}(W)) > 0$. Also $m(\tau(U) \times W) = n(\tau^{-1}(\tau(U)) \cap f'^{-1}(W))$ by assumption, so $n(\{i|f'(i) \notin F_g(i)\}) \geq m(\tau(U) \times W) > 0$ and therefore $f' \notin V_g$.

<u>Case 2</u>: Assume that $n(I_g(f)) > 0$ Suppose first that there is a player $i \in I_g(f)$ such that $n(i) > 0$. Then, by (13), $m(\{\tau(i)\} \times \{f(i)\}) \geq n(\{i\}) > 0$. Thus, by (14) and (15), $n(I_g(f')) \geq n(\tau^{-1}(\tau(i)) \cap f'^{-1}(f(i))) = m(\{\tau(i)\} \times \{f(i)\}) > 0$.

If there is no such player i, then the proof of Proposition 2 shows that there are open sets $(W \cap X) \subseteq \underline{N}$ and $Z \subseteq S$ such that $n(W \cap X \cap f^{-1}(Z)) > 0$ and $W \cap X \cap f^{-1}(Z) \subseteq I_g(f)$. By an argument analogous to Case 1, $n(W \cap X \cap f'^{-1}(Z)) > 0$ and $W \cap X \cap f'^{-1}(Z) \subseteq I_g(f')$ as well.

This argument has shown that, under the hypothesis of the lemma, $f \in E_g \Rightarrow f' \in E_g$. Since f and f' are interchangeable in the statement of the lemma, the converse implication holds as well. Q.E.D.

## 8. VARIATION OF THE PLAYERS OF A GAME

On a type space T (with continuous mass-revealing function b) and a strategy space S, consider a preference relation $\underline{P} \subseteq T \times (S \times M(D))^2$ and a feasible-strategy correspondence $\underline{F}:T \to S$. The pair $(\underline{P},\underline{F})$ will be called an <u>anonymous game form</u> if it satisfies

$$\underline{P} \text{ is open, and} \tag{17}$$

$$\underline{F} \text{ is closed and l.h.c.} \tag{18}$$



For any measure in $\underline{G}$, $\underline{P}$ and $\underline{F}$ will determine an anonymous strategic-form game in a natural way. Thus an anonymous game form can serve as a basis for studying the type of question that was described in the Introduction.

<u>Proposition 3</u>: Let $(\underline{P},\underline{F})$ be an anonymous game form, and let $n \in \underline{G}$. Then there is an anonymous game g having n as the masure of its players, and of which the preference relation $P_g$ and the feasible-strategy correspondence $F_g$ are defined by (14) and (15), respectively.

<u>Proof</u>: It must be shown that $P_g$ is open, and that $F_g$ is closed and l.h.c.

To show that $P_g$ is open, define $\pi:\underline{N} \times (S \times \underline{V})^2 \to T \times (S \times M(D))^2$ by $\pi(i,s',f',s,f) = (\tau(i),s',\mu(n,f'),s,\mu(n,f))$. It will be proved below, as Lemma 4, that $\mu$ is continuous in f when $\underline{V}$ has the topology of convergence in n-measure. By (14), $P_g = \pi^{-1}(\underline{P})$. This inverse image is open by (17), Lemma 4, and the continuity of $\tau$.

To show that $F_g$ is closed, define $\delta:\underline{N} \times S \to T \times S$ by $\delta(i,s) = (\tau(i),s)$. $F_g = \delta^{-1}(\underline{F})$ by (15), and this inverse image is closed by (18) and the continuity of $\delta$.

If U is open in S, then $\{i \mid F_g(i) \cap U \neq \emptyset\} = \tau^{-1}(\{t \mid \underline{F}(t) \cap U \neq \emptyset\})$, which is open by (18) and continuity of $\tau$. Thus $F_g$ is l.h.c. Q.E.D.

<u>Lemma 4</u>: Let $n \in \underline{G}$, and topologize $\underline{V}$ by convergence in $n$-measure. Then $\mu(n,f)$ is continuous as a function of $f$.

<u>Proof</u>: $\underline{V}$ is metrizable by [4, Lemma III.2.7], so sequential continuity implies continuity of $\mu$ with respect to $f$. Sequential continuity follows from [4, Corollary III.6.13] (Convergence in measure implies convergence almost everywhere) and [4, Corollary III.6.16] (Lebesgue's dominated convergence theorem). Q.E.D.

<u>Remark 2</u>: Proposition 3 formalizes the idea that the rules of a game may be defined without specifying the set of players (i.e., the support of n) or even the number of players who will participate. Such institutions as elections and auctions fit this description.

There is an evident analogy between an anonymous game form (in strategic form) and a topological family of pseudogames. The equilibrium correspondence of an anonymous game form would map $\underline{G}$ to $\underline{V}$. This correspondence cannot directly be characterized topologically because $\underline{V}$ is topologized in a way which depends on the measure $n \in \underline{G}$ of players. However, it is possible to study a correspondence which assigns $\mu(\{n\} \times E_g)$ to n, where g is the pseudogame determined by (14) and (15). Lemma 3 has shown that this correspondence completely determines the equilibrium correspondence of $(\underline{P},\underline{F})$. This determining correspondence will now be shown to be the equilibrium correspondence of a topological family derived from $(\underline{P},\underline{F})$.



## 9. THE STATISTICAL IMAGE OF AN ANONYMOUS GAME FORM

The <u>statistical image</u> of an anonymous game form $(\underline{P},\underline{F})$ is a topological family of pseudogames (G,T,S,V,P,F,A,J). T and S are the type and strategy spaces of $(\underline{P},\underline{F})$. G is the set of measures in M(T) which are the distribution of $\tau$ for some measure in $\underline{G}$. I.e., G is the set of $g \in M(T)$ which satisfy

$$\forall B \subseteq T \quad g(B) = n(\tau^{-1}(B)) \tag{19}$$

for some $n \in \underline{G}$. V is the set of distributions of decisions in strategy vectors of games in $(\underline{P},\underline{F})$. I.e., $V \subseteq M(D)$ is the image of $\underline{G} \times \underline{V}$ under $\mu$. P,F,A, and J are defined by

$$(g,t,c',c) \in P \iff (t,c',c) \in \underline{P} , \tag{20}$$

$$v \in F(g) \iff [\mathrm{supp}(v) \subseteq \underline{F} \text{ and } \forall B \subseteq T \quad v(B \times S) = g(B)], \tag{21}$$

$$(g,t,[s',v'],[s,v]) \in A \iff [(t,s') \in \underline{F} \text{ and} \tag{22}$$

$$\forall B \subseteq D \; v'(B) = v(B) + b(t)(\chi_B(t,s') - \chi_B(t,s))],$$

and

$$J(g,v) = \mathrm{supp}(v). \tag{23}$$

<u>Proposition 4</u>: The statistical image of an anonymous game form is a topological family.

Conditions (5)-(8) must be verified to prove Proposition 4. This verification requires some knowledge about open sets in $M(D)$, which is now given in Lemma 5. For $X$ a separable metric space, $U$ open in $X$ and $r \in \mathbb{R}$, define $W^+(U,r) = \{m \in M(X) | m(U) > r\}$ and $W^-(U,r) = \{m \in M(X) | m(cl(U)) < r\}$.

<u>Lemma 5</u>: The sets $W^+(U,r)$ and $W^-(U,r)$ are open in $M(X)$.

<u>Proof</u>: To show that $W^+(U,r)$ is open, suppose that $m \in W^+(U,r)$. A subbasic open set $W(f,q,q')$ will be found such that $m \in W(f,q,q') \subseteq W^+(U,r)$. Since $X$ is a separable metric space, there exist closed sets $H_1 \subseteq H_2 \subseteq \ldots$ such that $\bigcup_{k=1}^{\infty} H_k = U$. By Urysohn's lemma [4, Theorem I.5.2], for each $k$ there is a continuous function $f_k : X \to \mathbb{R}$ such that $\chi_{H_k} \le f_k \le \chi_U$. By Lebesque's dominated convergence theorem [4, Corollary III.6.16], $\int_X f_k dm > r$ for some $k$. Then $m \in W(f_k, \int_X f_k dm, \int_X f_k dm - r) \subseteq W^+(U,r)$.

Suppose now that $m \in W^-(U,r)$. Define $q = (r - m(cl(U)))/2$. Then $m \in W(\chi_X, m(X), q) \cap W^+(X \backslash cl(U), m(X \backslash cl(U)) - q) \subseteq W^-(U,r)$.

Q.E.D.



<u>Proof of Proposition 4</u>: $P = G \times \underline{P}$ by (20). This is open in $G \times T \times C^2$ by (17), so (5) holds.

To verify (6), it must be shown that if $v \notin F(g)$, then some neighborhood of $(g,v)$ in $G \times V$ is disjoint from $F$ (i.e., from the graph of $F$). By (21), $F = \{(g',v') \in G \times V | supp(v') \subseteq \underline{F}\} \cap [\bigcap_{B \subseteq T} \{(g',v') \in G \times V | v'(B \times S) = g'(B)\}]$.

First suppose that $supp(v) \not\subseteq \underline{F}$. Then $D \backslash \underline{F}$ is open by (18) and $v(D \backslash \underline{F}) > 0$, so $(G \times W^+(D \backslash \underline{F}, 0)) \cap F = \emptyset$.

Second, suppose that $v(B \times S) \neq g(B)$. To begin, suppose that $v(B \times S) > g(B)$. By regularity of $g$, there is an open $U \subseteq T$ with $B \subseteq U$ and $g(U) < v(B \times S) \le v(U \times S)$. Since $T$ is a separable metric space, there is a closed $H \subseteq U$ such that $g(U) < v(H \times S)$. By Urysohn's lemma, there is a continuous $f: T \to \mathbb{R}$ with $\chi_H \le f \le \chi_U$. Define $f': D \to \mathbb{R}$ by $f'(t,s) = f(t)$, and define $r > 0$ by $r = (\int_D f' dv - \int_T f dg)/2$. Then $(g,v) \in W(f, \int_T f dg, r) \times W(f', \int_D f' dv, r) \subseteq (M(T) \times M(D)) \backslash F$, since $\int_T f dg = \int_D f' dv$ if $v \in F(g)$.

Alternatively, suppose that $v(B \times S) < g(B)$. Then either $v((T \backslash B) \times S) > g(T \backslash B)$ (in which case it has just been shown that $(g,v) \notin cl(F)$), or else $v(D) < g(T)$ (in which case $W^+(T, (g(T) + v(D))/2) \times W^-(D, (g(T) + v(D))/2) \cap F = \emptyset$). Thus $(g,v) \notin cl(F)$ if $v(B \times S) \neq g(B)$, establishing (6).

Now it will be shown that $A$ is l.h.c. Suppose that $U' \subseteq S$ is open, that $W(f,q,r) \subseteq M(D)$, and that $(s',v') \in A(g,t,[s,v]) \cap (U' \times W(f,q,r))$.

Assume without loss of generality that $\int_D f dv' = q$. Note that $s' \in \underline{F}(t)$ by (22). If $X_U = \{x \in T | \underline{F}(x) \cap U \neq \emptyset\}$ for $U \subseteq U'$ open, then $X_U$ is open by (18). For $x \in X_U$, $u \in U$, $y \in S$ and $m \in W(v, \int_D f dv, r/2)$,

$$|[\int_D f dm + b(x)(f(x,u) - f(x,y))] - \int_D f dv'|$$

$$\leq r/2 + |b(x)(f(x,u) - f(x,y)) - b(t)(f(t,s') - f(t,s))|. \quad (24)$$

Neighborhoods $U \subseteq U'$ of $s'$, $X \subseteq X_U$ of $t$ and $Y \subseteq S$ of $y$ can be chosen so that the right hand side of (24) is less than $r$ on $W(v, \int_D f dv, r/2) \times X \times U \times Y$, by continuity of $b$ and $f$. Thus, for any $(g^o, t^o, s^o, v^o) \in G \times X \times Y \times (W(v, \int_D f dv, r/2) \cap V)$, $A(g^o, t^o, s^o, v^o) \cap (U' \times W(f,q,r)) = \emptyset$. I.e., (7) holds.

To show that $J$ is l.h.c., let $U \subseteq D$ be open and suppose that $d \in J(g,v) \cap U$. Since $d \in \mathrm{supp}(v)$, $v(U) > 0$, so $v \in W^+(U,0)$. Also, if $m(U) > 0$ for any $m \in M(D)$, then $\mathrm{supp}(m) \cap U \neq \emptyset$. Thus $G \times (W^+(U,0) \cap V) \subseteq \{(g',v') | F(g',v') \cap U \neq \emptyset\}$, so (8) is established.
Q.E.D.

If $n \in \underline{G}$ and $g \in G$ are related by (19), then $g$ is naturally associated with the game in $(\underline{P}, \underline{F})$ determined by $n$. The equivalence between the equilibria of these two pseudogames will be proved as Theorem 2. The proof requires several lemmas.

<u>Lemma 6</u>: If $g \in G$, then there are measures $g_0$ and $g_1$ in $M(T)$ such



that (i) $g_0$ is nonatomic and $\mathrm{supp}(g_0) \subseteq b^{-1}(\{0\})$, (ii) there is a sequence (possibly with repetitions) $t_1, t_2, \ldots$ in $T$ such that, for all $B \subset T$, $g_1(B) = \sum_{k=1}^{\infty} b(t_k)\chi_B(t_k)$, and (iii) $g = g_0 + g_1$.

<u>Proof</u>: Since $g \in G$, there is a measure $n \in \underline{G}$ such that $n$ and $g$ satisfy (19). By (12), $n$ has no atoms in $b^{-1}(\{0\}) \times [0,1]$, and $\mathrm{supp}(n) \cap b^{-1}((0,1])$ is a countable set. Let $i_1, i_2, \ldots$ be an enumeration of this set. For all $k$, $n(\{i_k\}) = b(\tau(i_k))$. Define $t_k = \tau(i_k)$ and let $g_1$ be defined by (ii). Define $g_0 = g - g_1$, so that (i) and (iii) hold. Q.E.D.

<u>Lemma 7</u>: If $v \in V$, then there are measures $v_0$ and $v_1$ in $M(D)$ such that (i) $v_0$ is nonatomic and $\mathrm{supp}(v_0) \subseteq b^{-1}(\{0\}) \times S$, (ii) there is a sequence (possibly with repetitions) $d_1, d_2, \ldots$ in $D$ such that, if $d_k = (t_k, s_k)$, $\forall B \subseteq D$ $v_1(B) = \sum_{k=1}^{\infty} b(t_k)\chi_B(d_k)$, and (iii) $v = v_0 + v_1$.

<u>Proof</u>: Let $v = \mu(n,f)$, and let $i_1, i_2, \ldots$ be the atoms of $n$. Define $t_k = \tau(i_k)$ and $s_k = f(i_k)$. Then $v_1$ is defined by (ii), and (i) and (iii) are satisfied by $v_0 = v - v_1$. Q.E.D.

<u>Lemma 8</u>: If $v \in F(g)$, then there exist $n \in \underline{G}$ and $f \in \underline{V}$ such that (i) $n$ and $g$ satisfy (19), and (ii) $v = \mu(n,f)$. In fact, a function $f$

satisfying (ii) exists for <u>any</u> n satisfying (i).

<u>Proof</u>: If $t_1, t_2, \ldots$ is the sequence described in Lemma 6, then by (21) there must be $s_1, s_2, \ldots$ such that $(t_1, s_1), (t_2, s_2), \ldots$ is the sequence described in Lemma 7. Define $n_1 \varepsilon M(\underline{N})$ by

$\forall B \subseteq \underline{N}\ n_1(B) = \sum_{k=1}^{\infty} b(t_k)\chi_B(t_k, 1/k)$. Define $n_0 \varepsilon M(\underline{N})$ by

$\forall X \subseteq T\ \forall Y \subseteq [0,1]\ n_0(X \times Y) = g_0(X) \cdot \lambda(Y)$, where $g_0$ is as in Lemma 6. Then $n = n_0 - n_1$ satisfies (i). To define f, first set $f(t_k, 1/k) = s_k$. Second, define a function $h: \underline{N} \to S$ such that $v_0 = \mu(n_0, h)$ by the method used in the proof of [7, Proposition II.2.2.6] (originally proved by Hart, Hildenbrand and Kohlberg [6] to assign an allocation to a distribution), and set $f = h$ on $b^{-1}(\{0\})$. Now f has been defined everywhere on supp(n), so (ii) is satisfied regardless of what value it is assigned elsewhere.

This construction of f works for any $n \varepsilon \underline{G}$ such that n and g satisfy (19). In particular, the proof appealed to in [6] requires only that $n_0$ be a nonatomic measure on $\underline{N}$, although it is stated only for $n_0 = g_0 \cdot \lambda$. However, the assumption that T and S are complete metric spaces is required in order for that proof to be applied. Q.E.D.

<u>Lemma 9</u>: If n and g satisfy (19), $(s', v') \varepsilon A(g, t, [s, v])$, $v = \mu(n, f)$, and $t \varepsilon$ supp(g), then there exists a player $i \varepsilon \tau^{-1}(t)$ for whom



$v' = \mu(n, a_i(s', f))$.

<u>Proof</u>: This follows from (10), (22) and (12). Q.E.D.

<u>Remark 3</u>: Crucial use has been made here of condition (12) defining $\underline{G}$. It has been necessary to define A in terms of the mass-revealing function b in order to assure lower hemicontinuity. Lemma 9 holds because (12) guarantees that this definition will be appropriate for almost all players.

<u>Theorem 2</u>: If $v \varepsilon V$, n and g satisfy (19), $E_g$ is defined by (16) for the game in $(\underline{P}, \underline{F})$ in which n is the measure of players, and E(g) is defined by (4) for the statistical image of $(\underline{P}, \underline{F})$, then the following are equivalent:

(i) $v \varepsilon E(g)$

(ii) $\exists f \varepsilon \underline{V}\ [f \varepsilon E_g \text{ and } v = \mu(n, f)]$

(iii) $\forall f \varepsilon \underline{V}\ [\mu(n, f) = v \Rightarrow f \varepsilon E_g]$.

<u>Proof</u>: That (i) and (ii) are equivalent is a consequence of Lemma 8, and of a comparison of (4) and (16) using Lemma 9. That (ii) and (iii) are equivalent follows from Lemma 3 and Lemma 8. Q.E.D.

## 10. THE EQUILIBRIUM CORRESPONDENCE OF AN ANONYMOUS GAME FORM

By the equilibrium correspondence of an anonymous game form is meant the equilibrium correspondence of its statistical image. The graph of this correspondence is now studied as a subset of $M(T) \times M(D)$. The first step is to describe $G$ as a subset of $M(T)$ and $V$ as a subset of $M(D)$.

Lemma 10: The measure $g \in M(T)$ is in $G$ if and only if $\forall t \in \mathrm{supp}(g)$ $[b(t) > 0 \Rightarrow g(\{t\})/b(t)$ is a strictly positive integer$]$. The measure $v \in M(D)$ is in $V$ if and only if $\forall (t,s) \in \mathrm{supp}(v)$ $[b(t) > 0 \Rightarrow v(\{(t,s)\})/b(t)$ is a strictly positive integer$]$.

Proof: Suppose that $\forall t \in \mathrm{supp}(g)$ $[b(t) > 0 \Rightarrow g(\{t\})/b(t)$ is a strictly positive integer$]$. Then there is an enumeration (possibly with repetitions) $t_1, t_2, \ldots$ of $b^{-1}((0,1]) \cap \mathrm{supp}(g)$ such that, for all $t \in b^{-1}((0,1])$, $\#\{k | t = t_k\} = g(\{t\})/b(t)$. Define $n_1 \in M(\underline{N})$ by

$n_1(X) = \sum_{k=1}^{\infty} b(t_k)\chi_X(t_k, 1/k)$. Define $g_0 \in M(T)$ by $g_0(B) = g(B \cap b^{-1}(\{0\}))$, define $n_0 \in M(\underline{N})$ by $\forall X \subseteq T\ \forall Y \subseteq [0,1]$ $n_0(X \times Y) = g_0(X) \cdot \lambda(Y)$, and define $n = n_0 + n_1$. Then $n \in \underline{G}$, and $n$ and $g$ satisfy (19), so $g \in G$. The converse implication follows from Lemma 6.

The proof of the equivalence for $v \in M(D)$ is analogous, using Lemma 7 and Lemma 8. Q.E.D.



Lemma 11: $G$ is closed in $M(T)$. $V$ is closed in $M(D)$.

Proof: Suppose that $m \in M(T)\backslash G$. By Lemma 10, for some $t \in b^{-1}((0,1]) \cap \mathrm{supp}(m)$, either $m(t) = 0$ or else $k < m(\{t\})/b(t) < k + 1$ for some integer $k$. If $m(\{t\}) = 0$, then there is a neighborhood $U$ of $t$ such that $m(\mathrm{cl}(U)) < \inf\{b(u) | u \in U\} = r$, by continuity of $b$, regularity of $m$ and normality of $T$ [4, Theorem I.6.3]. Thus $W^+(U,0) \cap W^-(U,r)$ is a neighborhood of $m$ disjoint from $G$, by Lemma 10.

Alternatively, suppose that $k < m(\{t\})/b(t) < k + 1$. Then there are real numbers $q < b(t) < r$ such that $kr < m(\{t\}) < (k + 1)q$. There is a neighborhood $U$ of $t$ with $U \subseteq b^{-1}((q,r))$ and $m(\mathrm{cl}(U)) < (k + 1)q$. Then $W^+(U,kr) \cap W^-(U,(k + 1)q)$ is a neighborhood of $m$ disjoint from $G$, by Lemma 10.

The proof that $V$ is closed in $M(D)$ is analogous. Q.E.D.

Theorem 3: The equilibrium correspondence of an anonymous game form is closed in $M(T) \times M(D)$. If $E$ is nonempty valued, $T$ is compact and $\underline{F}$ is u.h.c. and compact valued, then $E$ is upper hemicontinuous as a correspondence from $G$ to $V$.

Proof: The first assertion is immediate from Theorem 1, Theorem 2 and Lemma 11. The second assertion will be derived from the corollary of Theorem 1.

It may be assumed without loss of generality that $S = \underline{F}(T)$.

By [6, Chapter 1, Proposition B.III.3], S is then compact. Thus M(D) is the positive cone of C*(D), and hence is closed in C*(D). Since every measure in F(g) has the same total variation as g does, F is compact valued by Alaoglu's theorem [4, Corollary V.4.3].

It remains to be shown that F is u.h.c. By [6, Chapter 1, Theorem B.III.1], it is sufficient to show that if $g_1, g_2, \ldots$ is any sequence in G which converges to a measure g in G, and if $v_1, v_2, \ldots$ is any sequence in V such that $v_k \in F(g_k)$ for all k, then there is a strategy vector $v \in F(g)$ such that a subsequence of $v_1, v_2, \ldots$ converges to v. For sufficiently large k, $v_k(D) < g(T) + 1$. Thus, by Alaoglu's theorem, the tail of the sequence $v_1, v_2, \ldots$ lies in a compact subset of M(D), so a subsequence converges to a measure $v \in M(D)$. By Lemma 11, $v \in V$. By Theorem 2, $v \in F(g)$.

The upper hemicontinuity of E now follows from the corollary of Theorem 1. Q.E.D.

## 11. CONCLUSION: APPLICATION TO MATHEMATICAL ECONOMICS

Most of the institutions studied in economics are anonymous in character, and can be modeled as anonymous game forms. Often these institutions are not incentive compatible when they are populated by few agents, but they become so (i.e., their noncooperative equilibria achieve competitive allocations) asymptotically as the number of agents increases. This phenomenon has been studied by considering



sequences $n_1, n_2, \ldots$ of probability measures, such that each $n_k$ has finite support $H_k$ and $\#H_k \to \infty$, in the space $\underline{G}$ of an anonymous game form. The sequences which reflect the effects of the presence of many players, rather than of the change of players' types, are those which converge in the sense that they are uniformly tight [9, Chapter 2, Theorem 6.7] and that, for every neighborhood $U \subseteq T$, $\sum_{i \in H_k} \chi_U(\tau(i))/\#H_k$ converges. Given a sequence $f_1, f_2, \ldots$ in $\underline{V}$, such that $f_k$ is an equilibrium of the game with player space $n_k$, it may be asked (cf. [10]) whether the allocations resulting from these strategy vectors in their respective games are asymptotically competitive.

An alternative method of study has been developed in [3] and in [5]. Rather than studying the sequences just described directly, this method involves studying the equilibrium sets E(g) where $\mathrm{supp}(g) \subseteq b^{-1}(\{0\})$ in the statistical image of the anonymous game form. These measures g correspond via (19) to nonatomic measures in $\underline{G}$, among which the limits of the sequences $n_1, n_2, \ldots$ must lie. (N.B. limits are defined in terms of the measures $g_k \in G$ associated with $n_k$ by (19), since $\underline{G}$ itself is not closed. A limit in this sense must exist by [9, Chapter 1, Theorem 6.1, Theorem 6.7].)

Theorem 3 is used to infer the asymptotic competitiveness of the equilibrium outcomes of sequences of finite-player games, if all equilibrium allocations of nonatomic games in the anonymous game form are exactly competitive. This approach is convenient because (i) it removes the necessity to make explicit approximations, and (ii) the nontomicity of the player spaces dealt with can be exploited as in Proposition 2 here. However, it sacrifices the quantitative estimates of the divergence of finite-player noncooperative outcomes from competitive allocations which the direct approach provides.



REFERENCES

* I would like to thank Marcus Berliant, Donald Brown, Tatsuro Ichiishi, Ariel Rubenstein and Hugo Sonnenschein for their thoughtful comments on early drafts of this paper.

1. Aumann, R. J. "Markets with a Continuum of Traders," Econometrica, 32 (1964): 39-50.

2. Debreu, G. "A Social Equilibrium Existence Theorem," Proc. Nat. Acad. Sci. U.S.A., 38 (1952): 886-893.

3. Dubey, P., A. Mas-Colell and M. Shubik. "Efficiency Properties of Strategic Market Games: An Axiomatic Approach," Journal of Economic Theory, 22 (1980): 339-362.

4. Dunford, N. and J. T. Schwartz. Linear Operators, Part I. Interscience Publishers, New York, 1958.

5. Green, E. "Noncooperative Price taking in Large Dynamic Markets," Journal of Economic Theory, 22 (1980): 155-182.

6. Hart, S., W. Hildenbrand and E. Kohlberg. "On Equilibrium Allocations as Distributions on the Commodity Space," Journal of Mathematical Economics, 1 (1974): 159-166.

7. Hildenbrand, W. Core and Equilibria of a Large Economy. Princeton University Press, Princeton, 1974.

8. Kannai, Y. "Continuity Properties of the Core of a Market," _Econometrica_, 38 (1970): 791-815.

9. Parthasarathy, K. R. _Probability Measures on Metric Spaces_. Academic Press, New York, 1967.

10. Postlewaite, A. and D. Schmeidler. "Approximate Efficiency of Non-Walrasian Equilibria," _Econometrica_, 46 (1978): 127-135.

11. Roberts, K. "The Limit Points of Monopolistic Competition," _Journal of Economic Theory_, 22 (1980): 256-278.

12. Schmeidler, D. "Equilibrium Points of Nonatomic Games," _Journal of Statistical Physics_, 7 (1973): 295-300.